cloud cover, and in regions where the economics of the situation do not justify the establishment of permanent laboratory facilities.

References

[1] Stormer, C., *The Polar Auroras* (Oxford University Press, London, 1955), Chaps. 1–4, Part I, pp. 1–41, Figs. 1–191.

[2] Chamberlain, J. W., *Physics of the Aurora and Airglow* (Academic Press, New York, 1961), Chap. 4, pp. 100–150.

[3] *International Auroral Atlas* (Edinburgh University Press, Edinburgh, Great Britain, 1963).

[4] Davis, T. N., "The morphology of the polar aurora," J. Geophys. Res. **65**, 3497–3500 (1960); also "The morphology of auroral displays of 1957–1958, I and II," J. Geophys. Res. **67**, 59–74, 75–110 (1962).

[5] Sharp, R. D., Evans, J. E., Imhof, W. L., Johnson, R. G., Reagan, J. B., and Smith, R. V., "Particle measurements from a satellite over the northern auroral zone," Trans. Am. Geophys. Union **43**, 437 (1962).

[6] "Preliminary results from coordinated measurements on auroras," I. G. Bull. **77**, 1–11 (November 1963); also Trans. Am. Geophys. Union **44**, 1013–1083 (1963).

[7] Moe, C. R. and Tyer, A. A., "Auroral attenuation measurements on 20 and 40 Mc/s signals from a satellite beacon," Trans. Am. Geophys. Union **43**, 437 (1963).

JULY 1964 AIAA JOURNAL VOL. 2, NO. 7

Drag Displacements and Decay of Near-Circular Satellite Orbits

F. T. Geyling*
Bell Telephone Laboratories, Inc., Whippany, N. J.

A drag analysis is carried out for near-circular satellite orbits, as used in relay and reentry missions. This work is formulated in terms of moving coordinates that have been previously applied to perturbation studies involving lunisolar gravitation and radiation pressure. The equations of motion for the satellite are written in terms of displacement components relative to an unperturbed Keplerian or "nominal" orbit. These components form an orthogonal triad, whose origin always lies at the nominal satellite position on the elliptic path. The rationale of working from some reference orbit is common to all established techniques of celestial mechanics (Encke, Hansen, Vinti, von Zeipel, et al.). Although our formulation uses the most primitive form of reference trajectory, the calculations remain simple enough to permit an allowance for moderate eccentricity of the nominal orbit and variability of atmospheric conditions. Predictions of orbit altitude and inclination during "spiral decay" are also possible. Perturbations in satellite coordinates, as given here, have obvious applications in guidance work.

1. Introduction

In the analysis of orbital motion of some communication satellites and re-entry vehicles, a typical orbit possesses rather small eccentricities and such altitudes that the predominant perturbations are due to launch errors, the equatorial bulge, and atmospheric drag. The present paper applies a perturbation method that has previously yielded useful results in the study of extraterrestrial gravitation, radiation pressure, oblateness effects, and perturbations of the initial conditions.[1] Some applications of these results have also been made in the study of re-entry trajectories. The perturbative motion of the vehicle is described in terms of displacements from the instantaneous position that it would occupy on a nominal elliptic orbit in the absence of all disturbances. The practical advantage of this formulation for position forecasts and calculations of orbit decay is obvious. The need for such a prediction scheme with low eccentricity has been previously stated in the literature.[2]

If applied to a study of drag perturbations of nominally circular orbits, the computations based on the present method turn out to be surprisingly simple by comparison with the more challenging analyses based on other methods. Consequently, some allowance for the time dependence of atmospheric conditions can be made without appreciable difficulty. If, however, terms of $0(e)$ and $0(e^2)$ are included to account for some eccentricity of the nominal orbit, the algebra becomes more laborious. When rather large eccentricities are to be considered, an approach through Lagrange's planetary equations or by a suitable numerical scheme[3] may be more expedient.

2. Coordinate Systems and General Expressions for the Perturbations

Figure 1 illustrates the moving coordinate system used in the formulation of the present method. The perturbative displacement of the satellite from its nominal position $0'$ is described by the triad of components ξ, η, ζ. The angular argument by which $0'$ is located from the node on the nominal orbit shall be denoted by θ. We have $\theta = \omega + f$ where, for circular nominal orbits, ω marks a convenient reference point from which the independent variable f is measured and the latter increases linearly with time. For elliptic nominal orbits, ω, of course, denotes the nominal argument of perigee and f denotes the true anomaly. The initial position of $0'$ is given by f_0 at $t = t_0$. Once ξ, η, ζ have been found, the transformations from the orbital coordinates to the x, y, z system, and hence to the azimuth-elevation system for tracking radars, are quite straightforward and produce the pointing angles desired as the ultimate goal of this analysis.

Received April 18, 1963; revision received April 16, 1964. The material in this paper was presented orally at the 1960 International Astronautical Federation Congress in Stockholm. Other publications in this area[12–15] show that interest in this facet of satellite motion has persisted, and a written account of the author's effort at this time may not be reduced entirely to the role of a "historical note."

* Head, Analytical and Aerospace Mechanics. Member AIAA.

We obtain the differential equations for ξ, η, ζ to $0(e)$ in the eccentricity of the nominal orbit from Eqs. (11–13) of Ref. 1 with an obvious change of notation:

$$\xi'' - 2\eta' - 3\xi + 2e\xi \cos f - 2e(\xi' - \eta) \sin f = -(a^3/mk)(1 - 4e \cos f)\bar{V}_\xi \quad (1)$$

$$\eta'' + 2\xi' - e\eta \cos f - 2e(\eta' + \xi) \sin f = -(a^3/mk)(1 - 4e \cos f)\bar{V}_\eta \quad (2)$$

$$\zeta'' + \zeta - e\zeta \cos f - 2e\zeta' \sin f = -(a^3/mk)(1 - 4e \cos f)\bar{V}_\zeta \quad (3)$$

where k is the earth's mass times the gravitational constant, m is vehicle mass, a is semimajor axis of nominal orbit, $\bar{V}_{\xi,\eta,\zeta}$ are the partial derivatives of the perturbing potential V with respect to ξ, η, ζ, and the "primes" denote derivatives with respect to f.

To be consistent with the order of approximation in these equations, we may write the solution as

$$\xi = \bar{\xi} + eg \qquad \eta = \bar{\eta} + eh \qquad \zeta = \bar{\zeta} + ej \quad (4)$$

Extracting terms of $0(1)$ from Eqs. (1–3), we obtain the equations

$$\bar{\xi}'' - 2\bar{\eta}' - 3\bar{\xi} = -(a^3/mk)\bar{V}_\xi \quad (5)$$

$$\bar{\eta}'' + 2\bar{\xi}' = -(a^3/mk\bar{V}_\eta) \quad (6)$$

$$\bar{\zeta}'' + \bar{\zeta} = -(a^3/mk)\bar{V}_\zeta \quad (7)$$

If we denote a complementary and particular solution of Eqs. (5–7) by $\bar{\xi}_1$, $\bar{\eta}_1$, $\bar{\zeta}_1$ and $\bar{\xi}_2$, $\bar{\eta}_2$, $\bar{\zeta}_2$, then we have, from (17–22) in Ref 1,

$$\bar{\xi}_1 = 2\bar{\eta}_0' + 4\bar{\xi}_0 - (2\bar{\eta}_0' + 3\bar{\xi}_0) \cos f + \bar{\xi}_0' \sin f \quad (8)$$

$$\bar{\eta}_1 = \bar{\eta}_0 - 2\bar{\xi}_0' - 3(\bar{\eta}_0' + 2\bar{\xi}_0)f + 2(2\bar{\eta}_0' + 3\bar{\xi}_0) \sin f + 2\bar{\xi}_0' \cos f \quad (9)$$

$$\bar{\zeta}_1 = \bar{\zeta}_0' \sin f + \bar{\zeta}_0 \cos f \quad (10)$$

and

$$\bar{\xi}_2 = (a^3/mk)[-2\int\bar{V}_\eta df + 2 \cos f\int\bar{V}_\eta \cos f\, df + 2 \sin f\int\bar{V}_\eta \sin f\, df + \cos f\int\bar{V}_\xi \sin f\, df - \sin f\int\bar{V}_\xi \cos f\, df] \quad (11)$$

$$\bar{\eta}_2 = (a^3/mk)[3\int\int\bar{V}_\eta df\, df + 2\int\bar{V}_\xi df - 4 \sin f\int\bar{V}_\eta \cos f\, df + 4 \cos f\int\bar{V}_\eta \sin f\, df - 2 \sin f\int\bar{V}_\xi \sin f\, df - 2 \cos f\int\bar{V}_\xi \cos f\, df] \quad (12)$$

$$\bar{\zeta}_2 = (a^3/mk)[\cos f\int\bar{V}_\zeta \sin f df - \sin f\int\bar{V}_\zeta \cos f df] \quad (13)$$

where $\bar{\xi}_0$, $\bar{\eta}_0$, $\bar{\zeta}_0$ and $\bar{\xi}_0'$, $\bar{\eta}_0'$, $\bar{\zeta}_0'$ are the constants of integration arising from (5–7). Formulas (11–13) were obtained by variation of constants, and the integrals should run from the epoch at which $\bar{\xi}_0 \ldots \bar{\zeta}_0'$ are determined ($f = f_0$) to the instant for which ephemeris values are to be calculated.

Taking terms of $0(e)$ from (1–3), the differential equations for g, h, j are

$$g'' - 2h' - 3g = 2(2\bar{\xi}_0' - \bar{\eta}_0) \sin f - 4(\bar{\eta}_0' + 2\bar{\xi}_0) \cos f + 6(\bar{\eta}_0' + 2\bar{\xi}_0)f \sin f + 2(2\bar{\eta}_0' + 3\bar{\xi}_0) \cos 2f - 2\bar{\xi}_0' \sin 2f + 2(\bar{\xi}_2' - \bar{\eta}_2) \sin f - 2\bar{\xi}_2 \cos f + (4a^3/mk)\bar{V}_\xi \cos f \quad (14)$$

$$h'' + 2g' = -2(\bar{\eta}_0' + 2\bar{\xi}_0) \sin f + (\bar{\eta}_0 - 2\bar{\xi}_0') \cos f - 3(\bar{\eta}_0' + 2\bar{\xi}_0)f \cos f + 2(2\bar{\eta}_0' + 3\bar{\xi}_0) \sin 2f + 2\bar{\xi}_0' \cos 2f + 2(\bar{\eta}_2' + \bar{\xi}_2) \sin f + \bar{\eta}_2 \cos f + (4a^3/mk)\bar{V}_\eta \cos f \quad (15)$$

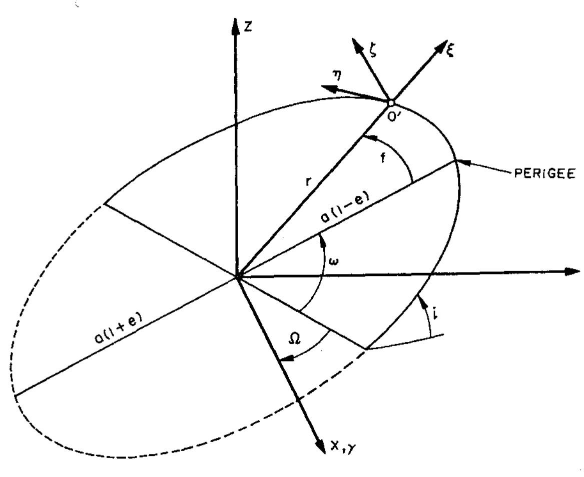

Fig. 1 Definitions of moving coordinates ξ, η, ζ.

$$j'' + j = -(\bar{\zeta}_0/2) + (3\bar{\zeta}_0'/2) \sin 2f + (3\bar{\zeta}_0/2) \cos 2f + 2\bar{\zeta}_2' \sin f + \bar{\zeta}_2 \cos f + (4a^3/mk)\bar{V}_\zeta \cos f \quad (16)$$

Since the differential operators in these equations are the same as in (5–7), their complementary solution may be absorbed in (8–10) without explicit distinction. A particular solution of (14–16) will be written as $g_{1,2,3}$, $h_{1,2,3}$, $j_{1,2,3}$, corresponding to the components of the right-hand sides involving $\bar{\xi}_0 \ldots \bar{\zeta}_0'$, $\bar{\xi}_2 \ldots \bar{\zeta}_2'$, or $\bar{V}_{\xi,\eta,\zeta}$, respectively. Thus,

$$g_1 = (\bar{\eta}_0 - 2\bar{\xi}_0') \sin f - \tfrac{5}{2}(\bar{\eta}_0' + 2\bar{\xi}_0) \cos f - 3(\bar{\eta}_0' + 2\bar{\xi}_0)f \sin f \quad (17)$$

$$h_1 = 7(\bar{\eta}_0' + 2\bar{\xi}_0) \sin f + (\bar{\eta}_0 - 2\bar{\xi}_0') \cos f - 3(\bar{\eta}_0' + 2\bar{\xi}_0)f \cos f - (\bar{\xi}_0'/2) \cos 2f - (\bar{\eta}_0' + \tfrac{3}{2}\bar{\xi}_0) \sin 2f \quad (18)$$

$$j_1 = -(\bar{\zeta}_0/2) - (\bar{\zeta}_0'/2) \sin 2f - (\bar{\zeta}_0/2) \cos 2f \quad (19)$$

and for g_2, h_2, j_2 and g_3, h_3, j_3, we replace $-(a^3/mk)\bar{V}_{\xi,\eta,\zeta}$ in (11–13) by

$$2(\bar{\xi}_2' - \bar{\eta}_2) \sin f - 2\bar{\xi}_2 \cos f$$

$$2(\bar{\eta}_2' + \bar{\xi}_2) \sin f + \bar{\eta}_2 \cos f$$

$$2\bar{\zeta}_2' \sin f + \bar{\zeta}_2 \cos f$$

and $(4a^3/mk)\bar{V}_{\xi,\eta,\zeta} \cos f$, respectively.

With the preceding differential equations and general expressions for their solution, we may now examine various aspects of drag perturbations in a sequence corresponding to their quantitative importance.

3. Perturbations of Circular Orbits Due to Atmospheric Resistance and Easterly Winds

The first and basic case to consider is an orbit with $e = 0$ where the disturbances stem from satellite drag in an atmosphere that performs diurnal rotations. If $\bar{v}$ is the satellite velocity relative to the rotating atmosphere, then the drag vector is $\mathbf{F}_A = -D\bar{v}^2 \boldsymbol{\varepsilon}_v$, where $\boldsymbol{\varepsilon}_v$ is the unit vector corresponding to $\bar{v}$, and $-D$ is drag force per (unit linear velocity)^2. The components of $\mathbf{F}_A$ in the η and ζ directions have the magnitudes $F_\eta = -D\bar{v}\bar{v}_\eta$ and $F_\zeta = -D\bar{v}\bar{v}_\zeta$, where $\bar{v}_\eta$ and $\bar{v}_\zeta$ are the components of $\bar{v}$ in the η and ζ directions.

The satellite velocity in an inertial frame has the circular value $(k/r_c)^{1/2}$, which points, of course, in the η direction, where r_c is the radius of the orbit. To obtain the satellite velocity relative to the atmosphere, we consider the earth's rotational wind velocity at orbital altitude, which is $\sigma r_c \cos\varphi'$, where σ is the earth's angular rate of rotation, φ' is the geocentric latitude of the satellite's instantaneous position, and the atmosphere is assumed to rotate as a rigid body. Resolving this easterly velocity into its η and ζ components, we obtain $\sigma r_c \cos\varphi' \sin\beta = \sigma r_c \cos i$ in the η direction and $-\sigma r_c \cos\varphi' \cos\beta = -\sigma r_c \sin i \cos\theta$ in the ζ direction, where β is the instantaneous azimuth of the η direction, and the final forms of these expressions were obtained with the help of some spherical trigonometry. Using these results in the expressions for the relative velocity components, we have

$$\bar{v}_\eta = (k/r_c)^{1/2} - \sigma r_c \cos i$$

$$\bar{v}_\zeta = \sigma r_c \sin i \cos\theta$$

and

$$\bar{v} = [\bar{v}_\eta{}^2 + \bar{v}_\zeta{}^2]^{1/2}$$

Thus, finally,

$$F_\eta = -D[(k/r_c)^{1/2} - \sigma r_c \cos i](k/r_c)^{1/2} = F \qquad (20)$$

and

$$F_\zeta = -D\sigma \sin i (kr_c)^{1/2} \cos\theta = G \cos\theta \qquad (21)$$

where we have made the simplification $\bar{v} \simeq (k/r_c)^{1/2}$. These quantities now take the place of $-\tilde{V}_\eta$ and $-\tilde{V}_\zeta$, respectively, n (11–13), whereas $\tilde{V}_\xi = 0$. F and G represent constant factors.

The results of the calculations are

$$\bar{\xi}_2 = (2r_c{}^3/mk)Ff \qquad (22)$$

$$\bar{\eta}_2 = (r_c{}^3F/mk)[4 - \tfrac{3}{2}f^2] \qquad (23)$$

$$\bar{\zeta}_2 = (r_c{}^3G/4mk)[2f \sin\theta - \cos\theta] \qquad (24)$$

$\bar{\xi}_2$ indicates that the satellite trajectory spirals toward the earth at a constant rate dr/dt if the atmospheric density can be assumed constant in the neighborhood of r_c and as long as the cumulative perturbation remains within a range that justifies the neglect of $0(\xi^2/r_c{}^2)$ etc., as was done in deriving the linearized Eqs. (11–13) of Ref. 1. The result for $\bar{\eta}_2$ shows that the advance of the vehicle over the angular position given by the nominal circular motion is a second-degree function of time. We note that (22) and (23) are devoid of oscillatory terms and remember that they constitute only a particular solution of (5) and (6). The satisfaction of given initial conditions of the motion will, in general, introduce a nontrivial form of the complementary solution (8) and (9), which contributes periodic perturbations. In $\bar{\zeta}_2$, finally, the secular term is again of interest, since it describes a linear decrease of orbit inclination in time. We note from the geometry at the node that $[(1/r_c)\bar{\zeta}_2{}']_{\theta=0}^{\theta=2n\pi}$ gives the change in orbit inclination at the completion of the nth revolution. Division by $2n\pi$ then yields the rate of change of inclination of the mean orbit. This is found from (24) to be

$$di/df = r_c{}^2G/2mk \qquad (25)$$

where we remember that $G < 0$. The physical interpretation of this result stems from the perturbative torque exerted by the easterly winds on the orbit which causes the angular momentum vector (normal to the orbit plane) to precess toward the earth's North Pole. In fact, this simple physical picture can be used for an independent derivation of (25).

As a numerical example for (22–25), we take $r_c = 5000$ statute miles, $i = 50°$, and assume that the vehicle consists of a 100-ft mylar balloon, 0.0005 in. thick. The data for atmospheric density from the literature[4–7] lead to typical results for secular displacements due to drag in the order of $\Delta\xi = -0.5 \times 10^{-5}$ mile, $\Delta\eta = +2.5 \times 10^{-5}$ mile, and $\Delta i = -0.5 \times 10^{-9}$ deg/rev.

We thus have a set of results for the basic case of drag perturbations. It should be noted that the Magnus effect on a spinning satellite may be formulated as conveniently as the easterly winds if one needs to include this phenomenon, but a more important refinement of our results consists of an allowance for eccentricity of the nominal orbit to $0(e)$.

4. Drag Perturbations of Eccentric Orbits

We obtain the correction of the expressions for drag displacements to the first order in e by substituting (20) and (21) in place of $-\tilde{V}_\eta$ and $-\tilde{V}_\zeta$ and $\tilde{V}_\xi = 0$ into (14–16) and following the procedure for g_2, h_2, j_2 and g_3, h_3, j_3 given in Sec. 2. The results are

$$g_2 = (a^3F/2mk)[-3f^2 \sin f - 9f \cos f + \tfrac{29}{2} \sin f] \qquad (26)$$

$$h_2 = (a^3F/2mk)[-3f^2 \cos f + 22f \sin f + 41 \cos f] \qquad (27)$$

$$j_2 = (a^3G/4mk)[\tfrac{5}{2} \cos\omega - \tfrac{1}{6} \cos(\omega + 2f) - f \sin\omega - f \sin(\omega + 2f)] \qquad (28)$$

and

$$g_3 = (4a^3F/mk)[f \cos f - \tfrac{3}{2} \sin f] \qquad (29)$$

$$h_3 = -(8a^3F/mk)[2 \cos f + f \sin f] \qquad (30)$$

$$j_3 = (2a^3G/mk)[-\cos\omega + \tfrac{1}{3} \cos(\omega + 2f)] \qquad (31)$$

In obtaining these results for the disturbing forces (20) and (21) as derived for a circular orbit, we have ignored the fact that, to $0(e)$ for an elliptic nominal orbit, $F_\xi \neq 0$ and $F_{\eta,\zeta}$ are subject to periodic fluctuations that may presumably be incorporated by the method of Sec. 5. According to Eq. (4), the results (26–31) combine with (22–24), where r_c is replaced by a, to yield the drag perturbations of slightly elliptic orbits (apart from the terms in $\bar{\xi}_0 \ldots \bar{\zeta}_0{}'$). The reader can easily see how (1–3) may be derived from Ref. 1 to higher-order terms in e and solved to yield the higher-order corrections of the drag solution. The next step in our investigation, however, concerns itself with the possibility that the drag characteristics experience short-term variations in time, i.e., F and G are modified by some time-dependent factor.

5. Perturbations with Short-Period Variations in Drag Parameters

One cause for time dependence of the vehicle's drag characteristics is the differences in atmospheric conditions between the illuminated and the shaded side of the earth as well as the oblateness of the atmosphere. These effects may indeed be quite appreciable.[8–11] Another cause for changing conditons is the possibility of satellite tumbling with consequent variations in frontal area and drag coefficient. A third phenomenon is the effect of solar activity on atmospheric density and the dependence of the satellite's effective cross section on its skin charge, which in turn is a function of atmospheric charge density and magnetic storms. The latter finally depend on far-flung mechanisms of the solar system and clouds of particles from interplanetary space. Although all of these phenomena may produce changes in drag which are noticeable over just a few anomalistic periods or even fractions thereof, some of them cannot be expected to manifest any sort of periodicity within a forecasting interval of a few days. We can expect, however, that atmospheric conditions, considered as functions of the true anomaly in the nominal orbit, may be represented as trigonometric series in f over a given interval. If such a series is written for each of the three sources of time-dependence, the perturbations due to the triple product of these series may be constructed (in principle at least) by using the results for an elementary triple product, which can be exhibited in a general notation.

Suppose that we multiply the factor F by the "time function":

$$[1 + H_1 \sin(\alpha_1 + \lambda_1 f)][1 + H_2 \sin(\alpha_2 + \lambda_2 f)] \times [1 + H_3 \sin(\alpha_3 + \lambda_3 f)] \qquad (32)$$

where H_1, H_2, H_3 are amplitude parameters; α_1, α_2, α_3 are phase angles; and λ_1, λ_2, λ_3 are frequency ratios between atmospheric conditions and the anomalistic motion. Then we simplify the notation to $\theta_1 = \alpha_1 + \lambda_1 f$, etc., and insert this time function into (20) and (11–13), to produce the following results:

$$\xi_2 = \frac{2a^3F}{mk}\left\{f - \sum_{i=1}^{3}\frac{H_i\cos\theta_i}{\lambda_i(1-\lambda_i^2)} + \frac{1}{2}\sum_{i\neq j}^{3}\sum^{3}\frac{H_iH_j\sin(\theta_i-\theta_j)}{(\lambda_i-\lambda_j)[1-(\lambda_i-\lambda_j)^2]} - \frac{1}{2}\sum_{i\neq j}^{3}\sum^{3}\frac{H_iH_j\sin(\theta_i+\theta_j)}{(\lambda_i+\lambda_j)[1-(\lambda_i+\lambda_j)^2]} + \frac{1}{4}H_1H_2H_3[K(\theta_1,\theta_2,\theta_3,\lambda_1,\lambda_2,\lambda_3) - K(-\theta_1\ldots-\lambda_1\ldots) - K(.-\theta_2\ldots-\lambda_2.) - K(\ldots-\theta_3\ldots-\lambda_3)]\right\} \quad (33)$$

where

$$K(\theta_1,\theta_2,\theta_3,\lambda_1,\lambda_2,\lambda_3) = \frac{\cos(\theta_1+\theta_2+\theta_3)}{(\lambda_1+\lambda_2+\lambda_3)[1-(\lambda_1+\lambda_2+\lambda_3)^2]}$$

$$\bar{\eta}_2 = \frac{a^3F}{mk}\left\{4 - \frac{3}{2}f^2 + \sum_{i=1}^{3}\frac{H_i(3+\lambda_i^2)\sin\theta_i}{\lambda_i^2(1-\lambda_i^2)} + \frac{1}{2}\sum_{i\neq j}^{3}\sum^{3}\frac{H_iH_j[3+(\lambda_i-\lambda_j)^2]\cos(\theta_i-\theta_j)}{(\lambda_i-\lambda_j)^2[1-(\lambda_i-\lambda_j)^2]} - \frac{1}{2}\sum_{i\neq j}^{3}\sum^{3}\frac{H_iH_j[3+(\lambda_i+\lambda_j)^2]\cos(\theta_i+\theta_j)}{(\lambda_i+\lambda_j)^2[1-(\lambda_i+\lambda_j)^2]} + \frac{1}{4}H_1H_2H_3[-L(\theta_1,\theta_2,\theta_3,\lambda_1,\lambda_2,\lambda_3) + L(-\theta_1\ldots-\lambda_1\ldots) + L(.-\theta_2\ldots-\lambda_2.) + L(\ldots-\theta_3\ldots-\lambda_3)]\right\} \quad (34)$$

where

$$L(\theta_1,\theta_2,\theta_3,\lambda_1,\lambda_2,\lambda_3) = \frac{[3+(\lambda_1+\lambda_2+\lambda_3)^2]\sin(\theta_1+\theta_2+\theta_3)}{(\lambda_1+\lambda_2+\lambda_3)^2[1-(\lambda_1+\lambda_2+\lambda_3)^2]}$$

The occurrences of H_1, H_2, H_3 in the individual terms of these results indicate the product terms of (32) from which they originated. Analogous results can be derived for $\bar{\zeta}_2$ by introducing a time function like (32) into (21).

This completes our discussion of first-order perturbations. Further refinements, in order of quantitative importance, would probably be the inclusion of terms to $0(e^2)$, as suggested in Sec. 4, and the consideration of second-order perturbations. In the latter case, one would also need to consider the cross coupling between drag perturbations and other physical factors such as oblateness.

6. Drag Decay of Near-Circular Orbits

Besides the forecasting of vehicle positions over a few orbital periods or fractions thereof, we may use the first-order analysis to estimate orbit decay. As we know, the decay and final collapse of an elliptic satellite orbit is usually subdivided into four phases. In the first phase, the perigee altitude decreases only at a slow rate, while apogee moves in toward the earth much more rapidly. Thus, with a simultaneous loss in major axis and eccentricity, a circular orbit is being approached. An exact circle is, of course, never obtained because of the everpresent drag perturbation, but when apogee and perigee cease to be distinguishable, the orbit degenerates into a spiral. This is the second stage, in which the altitude throughout the orbit decreases much more rapidly than the perigee height during the first phase. This is because appreciable drag is now encountered on all parts of the orbit, which keeps increasing as the orbit shrinks and brings the vehicle into denser atmospheric strata. The third stage forms the transition from the spiral stage to the satellite's final plunge. During this interval, the satellite's motion is no longer near-horizontal, but the angle of descent increases at an appreciable rate. The total arc, which is swept out in this phase, hardly covers more than one revolution. In the fourth and final stage, the satellite, if it has not burnt up before that time, approaches the earth in near-vertical fall. For communication or observation relays, the third and fourth phases are of no practical value, whereas a near-circular launch causes drag decay to commence with the spiral stage. We shall therefore limit our study to this phase.

Writing out (22), we find

$$\xi = -(2D/m)[(kr_c)^{1/2} - \sigma r_c^2\cos i]t$$

where we have substituted $f = (k/r_c^3)^{1/2}t$, and r_c is now a function of t. D contains t explicitly because of long-term variations in atmospheric conditions and implicitly because of the dependence of ambient density on r_c. The fact that the atmosphere does not rotate as a rigid body could be expressed by making σ a function of r_c, and the calculations could also be modified to include time dependence of i. Since, however, the diurnal term in ξ is of rather small magnitude to begin with, and an approximation was already made in deriving (22), these added refinements are hardly justified.

From ξ we obtain

$$\frac{d\xi}{dt} = \frac{dr_c}{dt} = \frac{-(2/m)[(kr_c)^{1/2} - \sigma r_c^2\cos i][d(Dt)/dt]}{\{1 + (2Dt/m)[(k/2r_c^{1/2}) - 2\sigma r_c\cos i]\}} \quad (35)$$

from which we find

$$r_c = r_{c_0} + \int_{t_0}^{t}\frac{dr_c}{dt}\,dt$$

by numerical integration. Similarly, we compute from (25)

$$i = i_0 - \frac{\sigma}{2m}\int_{t_0}^{t} Dr_c\sin i\,dt \quad (36)$$

where we employ the results of (35).

Analytic representations of atmospheric density against altitude have been repeatedly obtained in the past by piecewise fitting of exponential functions to experimental data. A variety of expressions may be suggested for this purpose, which also make allowance for atmospheric stratification.

7. Conclusions

The foregoing sections have illustrated the application of a method of moving coordinates to orbital predictions accounting for drag over a short range or to long-range estimates of orbit decay. It should be kept in mind that the results of Secs. 3–5 are functions of the true anomaly of the nominal orbit, which may be slightly elliptic. This presents certain implications about the use of these results.

The expressions for the perturbations were given as indefinite integrals in f, implying that the formulas (8–34) may be evaluated between any two values of f to yield the cumulative perturbations in the corresponding time interval. In a case where the nominal orbit is as yet to be established and a position forecast is to be made from smoothed data for satellite position and velocity, one may use the state variables derived from these observations to calculate the nominal orbit elements. If in this process the eccentricity appears too small for an accurate determination of ω by conventional means, we assume a circular nominal orbit through the initial position of the vehicle and let the actual noncircularity be absorbed by the integration constants $\bar{\xi}_0, \ldots, \bar{\zeta}_0'$ of the perturbations. To be more explicit, we solve the simultaneous system of equations

$$\begin{aligned} [\bar{\xi}_1 + \bar{\xi}_2]_{\theta_0} &= 0 \\ &\vdots \\ n[\bar{\xi}_1' + \bar{\xi}_2']_{\theta_0} &= \dot{X} \\ n[\bar{\eta}_1' + \bar{\eta}_2']_{\theta_0} &= v - (k/X)^{1/2} \\ [\bar{\zeta}_1' + \bar{\zeta}_2']_{\theta_0} &= 0 \end{aligned} \tag{37}$$

where

n = $(k/X^3)^{1/2}$, the angular rate for the nominal orbit
X = magnitude of the initial radius vector
$\dot{X}$ = initial radial velocity
v = initial tangential velocity
θ_0 = $\omega + f_0$, the initial angle from node to vehicle

For orbits with small inclination, and, hence, ill-defined nodes, we cease to use θ and θ_0, but measure the nominal position of the vehicle from some angular reference in the fundamental (xy) plane.

References

[1] Geyling, F. T., "Satellite perturbations from extraterrestrial gravitation and radiation pressure," J. Franklin Inst. **269**, 375 (May 1960).

[2] Musen, P., "Application of Hansen's theory to the motion of an artificial satellite in the gravitational field of the earth," J. Geophys. Res. **64**, 2271 (1959).

[3] Woolley, R.v.d. R., "Planetary coordinates for 1960–1980," *Her Majesty's Nautical Almanac Office* (Her Majesty's Stationery Office, London, 1958); descriptions of classical astronomical perturbation methods on p. 145.

[4] Sterne, T. E., "High altitude atmospheric density," Phys. Fluids **1**, 165 (1958).

[5] Minzner, R. A., Ripley, W. S., and Condon, T. P., *U. S. Extension to the ICAO Standard Atmosphere of 1956* (U. S. Government Printing Office, Washington, D. C., 1958).

[6] Minzner, R. A., Champion, K. S. W., and Pond, H. L., "The ARDC model atmosphere, 1959," Airforce Surveys in Geophysics, no. 115, Airforce Cambridge Research Center (August 1959).

[7] "Project Mercury, a composite model atmosphere for use in project Mercury Calculations," NASA Langley Research Center, Project Mercury Working Paper 205 (September 29, 1961).

[8] Parkyn, D. G., "Atmospheric tides and earth satellite observations," Nature **183**, 1045 (April 1959).

[9] Jones, L. M., Fischbach, F. F., and Peterson, J. W., "Seasonal and latitude variations in upper-air density," International Geophysical Year Rocket Rept., Ser. 1, International Geophysical Year World Data Center (July 1958).

[10] Sheppard, P. A., "Dynamics of the upper atmosphere," J. Geophys. Res. **64**, 2116 (1959).

[11] Jacchia, L. G., "Solar effects on the acceleration of artificial satellites," Smithsonian Astrophysical Observatory, Special Rept. 29 (September 1959).

[12] Perkins, F. M., "An analytical solution for flight time of satellites in eccentric and circular orbits," Astronaut. Acta **4**, 113 (1958).

[13] Zee, C. H., "Trajectories of satellites under the influence of air drag," AIAA Preprint 63-392 (1963).

[14] Forster, K. and Baker, R. M. L., "Orbit determination by linearized drag analysis," AIAA Preprint 63-428 (1963).

[15] Otterman, J. and Lichtenfeld, K., "Decay of apogee altitude, perigee altitude, and semimajor axis of satellite orbits of small eccentricity," General Electric Co., Technical Information Ser. 63 SD 227 (1963).